

Investigation of the dynamic behavior of a finite crack in the functionally graded materials by use of the Schmidt method

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Abstract

In this paper, the dynamic behavior of a finite crack in the functionally graded materials subjected to the harmonic stress waves is investigated by means of the Schmidt method. By use of the Fourier transform and defining the jumps of the displacements across the crack surfaces as the unknown functions, two pairs of dual integral equations are derived. To solve the dual integral equations, the jumps of the displacements across the crack surfaces are expanded in a series of Jacobi polynomials. Numerical examples are provided to show the effects of the crack length, the circular frequency of incident wave and the materials constants upon the stress intensity factor of the crack.

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1. Introduction

A new class of engineered materials namely, functionally gradient materials (FGMs) have been developed primarily for use in high temperature applications [1]. The composition in these FGMs, prepared using techniques like power metallurgy, chemical vapor deposition, centrifugal casting, etc., is graded along the thickness. The spatial variation of the material composition results in a medium with varying elastic and physical properties and calls for investigation into the fracture of FGMs under different loading conditions. In particular, the use of the graded material as interlayers in the bonded media is one of the highly effective and promising applications in eliminating various shortcoming resulting from stepwise property mismatch inherent in piecewise homogeneous composite media [2–4].

From the fracture mechanics viewpoint, the presence of a graded interlayer would play an important role in determining the crack driving forces and fracture resistance parameters. In an attempt to address the issues pertaining to the fracture analysis of bonded media with such transitional interfacial properties, a series of solutions to certain crack problems was obtained by Erdogan and co-workers [5–7]. Similar problems of delamination or an interface crack between the functionally graded coating and the substrate were considered in [8–10]. The dynamic crack problem for the non-homogeneous composite materials was considered in [11] but they considered the FGM layer as multi-layered homogeneous media. Relatively fewer experimental and numerical investigations of the fracture

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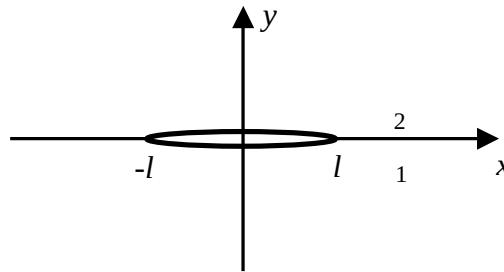


Fig. 1. Geometry of a finite crack in the functionally graded materials.

behavior of FGMs have been conducted. Experimental investigations on the fracture of FGMs are limited due to the high cost and elaborate facilities required for processing FGMs [12–14]. The finite element method has also been used to simulate the fracture behavior of cracked FGMs [15,16]. The crack problem in FGM layers under thermal stresses was studied by Erdogan and Wu [17]. They considered an unconstrained elastic layer under statically self-equilibrating thermal or residual stresses. The interface crack problem for a non-homogeneous coating bonded to a homogeneous substrate was investigated in [18]. Existing studies which consider the fracture behavior of FGMs are mainly limited to quasi-static problems. In this case, inertia effects do not play a role and can be ignored. However, it should be mentioned that most FGMs will be used in critical situations, where significant dynamic loading may be involved. The dynamic fracture behavior of FGMs has received little attention from the scientific community. Examples of dynamic analysis include the study of the steady-state dynamic crack propagation in an interphase with spatially varying elastic properties under anti-plane loading conditions reported in [19], and the steady-state dynamic fracture of FGMs under in-plane loading with the material properties being assumed to vary along the direction of crack propagation reported in [20]. Experimental studies of the dynamic fracture of FGMs with discrete property variation using photoelasticity technique were also conducted in [21]. The dynamic crack propagation problems were studied in [22,23]. In spite of these efforts, the understanding of the dynamic fracture process of FGMs is still limited.

In this paper, the dynamic behavior of a finite crack in the functionally graded materials subjected to the harmonic stress waves is investigated by means of the Schmidt method. It is also assumed that the elastic properties of FGMs spatially vary perpendicular to the plane of the crack. Young's modulus and mass density of the model are assumed to vary exponentially while Poisson's ratio remains constant. The analytical study is based on the use of the Fourier transform technique and a somewhat different approach, named as the Schmidt method [24,25]. To solve the dual integral equations, the jumps of the displacements across crack surfaces are expanded in a series of Jacobi polynomials. Numerical solutions are obtained for the stress intensity factors.

2. Formulation of the problem

It is assumed that there is a finite crack in the functionally graded materials as shown in Fig. 1. The lower half plane of the functionally graded materials is denoted as material 1. The upper half plane of the functionally graded materials is denoted as material 2. As discussed in [22], to make the analysis tractable, the elastic parameters $\mu(y)$, and $\rho(y)$ are approximated by

$$\mu(y) = \mu_0 e^{\beta y}, \quad \rho(y) = \rho_0 e^{\beta y}, \quad (1)$$

where β is a constant.

In this paper, the harmonic stress wave is vertically incident. Let ω be the circular frequency of the incident wave. $-\sigma_0$ is a magnitude of the incident wave. In what follows, the time dependence of all field quantities assumed to

be of the form $e^{-i\omega t}$ will be suppressed but understood. $\sigma_y^{(j)}(x, y, t)$, $\sigma_x^{(j)}(x, y, t)$ and $\tau_{xy}^{(j)}(x, y, t)$ (The superscript $j = 1, 2$ corresponds to the lower half plane and the upper half plane through in this paper.) represent the stress components, respectively. As discussed in [26], the boundary conditions of the present problem can be written as follows:

$$\sigma_y^{(1)}(x, 0, t) = \sigma_y^{(2)}(x, 0, t) = -\sigma_0, \quad \tau_{xy}^{(1)}(x, 0, t) = \tau_{xy}^{(2)}(x, 0, t) = 0, \quad |x| \leq l, \quad (2)$$

$$\sigma_y^{(1)}(x, 0, t) = \sigma_y^{(2)}(x, 0, t), \quad \tau_{xy}^{(1)}(x, 0, t) = \tau_{xy}^{(2)}(x, 0, t), \quad |x| > l, \quad (3)$$

$$u^{(1)}(x, 0, t) = u^{(2)}(x, 0, t), \quad v^{(1)}(x, 0, t) = v^{(2)}(x, 0, t), \quad |x| > l. \quad (4)$$

By denoting $u^{(j)}(x, y, t)$ and $v^{(j)}(x, y, t)$ as the displacement components in the x - and y -directions, respectively, the constitutive relations for the FGMs are written as

$$\sigma_x^{(j)}(x, y, t) = \frac{\mu_0 e^{\beta y}}{k-1} \left[(1+k) \frac{\partial u^{(j)}}{\partial x} + (3-k) \frac{\partial v^{(j)}}{\partial y} \right] \quad (j = 1, 2), \quad (5)$$

$$\sigma_y^{(j)}(x, y, t) = \frac{\mu_0 e^{\beta y}}{k-1} \left[(1+k) \frac{\partial v^{(j)}}{\partial y} + (3-k) \frac{\partial u^{(j)}}{\partial x} \right] \quad (j = 1, 2), \quad (6)$$

$$\tau_{xy}^{(j)}(x, y, t) = \mu_0 e^{\beta y} \left[\frac{\partial u^{(j)}}{\partial y} + \frac{\partial v^{(j)}}{\partial x} \right] \quad (j = 1, 2), \quad (7)$$

where $k = 3 - 4\nu$ for the state of plane strain, $k = (3 - \nu)/(1 + \nu)$ for the state of generalized plane stress. ν is the Poisson's ratio. The Poisson's ratio ν is taken to be a constant; owing to the fact its variation within a practical range has the rather insignificant influence on the value of the near-tip driving for fracture [5–7]. $\beta \neq 0$ for the functionally graded materials. When $\beta = 0$, it will return to the homogenous material case. In this paper, we just consider the plane strain problem.

In the absence of body forces, the elastic behavior of the medium with the variable shear modulus and the variable density in (1) is governed by following equations:

$$(1+k) \frac{\partial^2 u^{(j)}}{\partial x^2} + (k-1) \frac{\partial^2 u^{(j)}}{\partial y^2} + 2 \frac{\partial^2 v^{(j)}}{\partial x \partial y} + (k-1) \beta \left(\frac{\partial u^{(j)}}{\partial y} + \frac{\partial v^{(j)}}{\partial x} \right) = - \frac{(k-1) \rho_0 \omega^2}{\mu_0} u^{(j)}, \quad (8)$$

$$(1+k) \frac{\partial^2 v^{(j)}}{\partial y^2} + (k-1) \frac{\partial^2 v^{(j)}}{\partial x^2} + 2 \frac{\partial^2 u^{(j)}}{\partial x \partial y} + \beta \left[(1+k) \frac{\partial v^{(j)}}{\partial y} + (3-k) \frac{\partial u^{(j)}}{\partial x} \right] = - \frac{(k-1) \rho_0 \omega^2}{\mu_0} v^{(j)}. \quad (9)$$

3. Solution

Because of the symmetry, it suffices to consider the problem for $x \geq 0$, $|y| < \infty$. The system of above governing equations is solved, using the Fourier integral transform technique to obtain the general expressions for the displacement components as

$$u^{(1)} = \frac{2}{\pi} \int_0^\infty \sum_{i=1}^2 A_i(s) e^{-\lambda_i + 2y} \sin(sx) ds, \quad v^{(1)} = \frac{2}{\pi} \int_0^\infty \sum_{i=1}^2 m_{i+2}(s) A_i(s) e^{-\lambda_i + 2y} \cos(sx) ds, \quad (10)$$

$$u^{(2)} = \frac{2}{\pi} \int_0^\infty \sum_{i=1}^2 B_i(s) e^{-\lambda_i y} \sin(sx) ds, \quad v^{(2)} = \frac{2}{\pi} \int_0^\infty \sum_{i=1}^2 m_i(s) B_i(s) e^{-\lambda_i y} \cos(sx) ds \quad (11)$$

and from (5)–(7), the stress components are obtained as

$$\begin{aligned}\sigma_y^{(1)}(x, y) &= \frac{2\mu_0 e^{\beta y}}{\pi(k-1)} \int_0^\infty \sum_{i=1}^2 [-(k+1)m_{i+2}(s)\lambda_{i+2} + s(3-k)]A_i(s) e^{-\lambda_{i+2}y} \cos(sx) ds, \\ \tau_{xy}^{(1)}(x, y) &= \frac{2\mu_0 e^{\beta y}}{\pi} \int_0^\infty \sum_{i=1}^2 [-\lambda_{i+2} - m_{i+2}(s)s]A_i(s) e^{-\lambda_{i+2}y} \sin(sx) ds,\end{aligned}\quad (12)$$

$$\begin{aligned}\sigma_y^{(2)}(x, y) &= \frac{2\mu_0 e^{\beta y}}{\pi(k-1)} \int_0^\infty \sum_{i=1}^2 [-(k+1)m_i(s)\lambda_i + s(3-k)]B_i(s) e^{-\lambda_i y} \cos(sx) ds, \\ \tau_{xy}^{(2)}(x, y) &= \frac{2\mu_0 e^{\beta y}}{\pi} \int_0^\infty \sum_{i=1}^2 [-\lambda_i - m_i(s)s]B_i(s) e^{-\lambda_i y} \sin(sx) ds,\end{aligned}\quad (13)$$

where s is the transform variable. A_1, A_2, B_1 and B_2 are arbitrary unknowns, $\lambda_i(s)$ ($i = 1, 2, 3, 4$) are the roots of the characteristic equation

$$\lambda^4 - 2\lambda^3\beta + (\beta^2 - 2s^2)\lambda^2 + 2\beta s^2\lambda + s^4 + \frac{3-k}{k+1}\beta^2 s^2 + \frac{2k\rho_0\omega^2}{(k+1)\mu_0}(-s^2 + \lambda^2 - \beta\lambda) + \frac{k-1}{k+1}\left(\frac{\rho_0\omega^2}{\mu_0}\right)^2 = 0 \quad (14)$$

and $m_i(s)$ ($i = 1, 2, 3, 4$) are expressed for each root $\lambda_i(s)$ as

$$m_i(s) = \frac{-(k+1)s^2 + (k-1)\lambda_i^2 - \beta(k-1)\lambda_i}{-2s\lambda_i + s\beta(k-1)} \quad (15)$$

Eq. (14) can be rewritten as the following form

$$(\lambda^2 - \lambda\beta - s^2)^2 + \frac{3-k}{k+1}\beta^2 s^2 + \frac{2kc^2(\lambda^2 - \beta\lambda - s^2)}{k+1} + \frac{c^4(k-1)}{k+1} = 0, \quad (16)$$

where $c^2 = \rho_0\omega^2/\mu_0$.

The roots may be obtained as

$$\lambda_1 = \frac{\beta + \sqrt{\beta^2 - 4\left((kc^2/(k+1)) - s^2 - \sqrt{c^4/(k+1)^2 - s^2\beta^2(3-k)/(k+1)}\right)}}{2}, \quad (17)$$

$$\lambda_2 = \frac{\beta + \sqrt{\beta^2 - 4\left((kc^2/(k+1)) - s^2 + \sqrt{c^4/(k+1)^2 - s^2\beta^2(3-k)/(k+1)}\right)}}{2}, \quad (18)$$

$$\lambda_3 = \frac{\beta - \sqrt{\beta^2 - 4\left((kc^2/(k+1)) - s^2 - \sqrt{c^4/(k+1)^2 - s^2\beta^2(3-k)/(k+1)}\right)}}{2}, \quad (19)$$

$$\lambda_4 = \frac{\beta - \sqrt{\beta^2 - 4\left((kc^2/(k+1)) - s^2 + \sqrt{c^4/(k+1)^2 - s^2\beta^2(3-k)/(k+1)}\right)}}{2}. \quad (20)$$

From Eqs. (10)–(13), it can be seen that there are four unknown functions (in Fourier space they are functions of s), i.e., A_1 , A_2 , B_1 and B_2 which can be obtained from the boundary conditions. To solve the present problem, the jumps of the displacements across the crack surfaces can be defined as follows:

$$f_1(x) = u^{(2)}(x, 0) - u^{(1)}(x, 0), \quad (21)$$

$$f_2(x) = v^{(2)}(x, 0) - v^{(1)}(x, 0), \quad (22)$$

where $f_1(x)$ is an odd function and $f_2(x)$ an even function.

Applying the Fourier transforms and the boundary conditions (2)–(4), it can be obtained

$$[X_1] \begin{bmatrix} A_1(s) \\ A_2(s) \end{bmatrix} = [X_2] \begin{bmatrix} B_1(s) \\ B_2(s) \end{bmatrix}, \quad (23)$$

$$[X_3] \begin{bmatrix} B_1(s) \\ B_2(s) \end{bmatrix} - [X_4] \begin{bmatrix} A_1(s) \\ A_2(s) \end{bmatrix} = \begin{bmatrix} \bar{f}_1(s) \\ \bar{f}_2(s) \end{bmatrix}, \quad (24)$$

where the matrices $[X_i]$ ($i = 1, 2, 3, 4$) can be seen in [Appendix A](#).

A superposed bar indicates the Fourier transform through the paper. If $f(x)$ is an even function, the Fourier transform is defined as follows:

$$\bar{f}(s) = \int_0^\infty f(x) \cos(sx) dx, \quad f(x) = \frac{2}{\pi} \int_0^\infty \bar{f}(s) \cos(sx) ds. \quad (25)$$

If $f(x)$ is an odd function, the Fourier transform is defined as follows:

$$\bar{f}(s) = \int_0^\infty f(x) \sin(sx) dx, \quad f(x) = \frac{2}{\pi} \int_0^\infty \bar{f}(s) \sin(sx) ds. \quad (26)$$

By solving four Eqs. (23)–(24) with four unknown functions, substituting the solutions into Eq. (13) and applying the boundary conditions, it can be obtained

$$\sigma_y^{(2)}(x, 0, t) = \frac{2}{\pi} \int_0^\infty [d_1(s) \bar{f}_1(s) + d_2(s) \bar{f}_2(s)] \cos(sx) ds = -\sigma_0, \quad 0 \leq x \leq l, \quad (27)$$

$$\tau_{xy}^{(2)}(x, 0, t) = \frac{2}{\pi} \int_0^\infty [d_3(s) \bar{f}_1(s) + d_4(s) \bar{f}_2(s)] \sin(sx) ds = 0, \quad 0 \leq x \leq l, \quad (28)$$

$$\int_0^\infty \bar{f}_1(s) \sin(sx) ds = 0, \quad x > l, \quad (29)$$

$$\int_0^\infty \bar{f}_2(s) \cos(sx) ds = 0, \quad x > l, \quad (30)$$

where $d_1(s)$, $d_2(s)$, $d_3(s)$ and $d_4(s)$ are known functions, and can be seen in [Appendix A](#), respectively.

To determine the unknown functions $\bar{f}_1(s)$ and $\bar{f}_2(s)$, the above two pairs of dual integral equations (27)–(30) must be solved.

4. Solution of the dual integral equations

From the nature of the displacement along the crack line, it can be obtained that the jumps of the displacements across the crack surface are finite, differentiable and continuum functions. Hence, the jumps of the displacements across the crack surface can be expanded by the following series:

$$f_1(x) = \sum_{n=0}^{\infty} a_n P_{2n+1}^{(1/2, 1/2)} \left(\frac{x}{l} \right) \left(1 - \frac{x^2}{l^2} \right)^{1/2} \quad \text{for } 0 \leq x \leq l, \quad (31)$$

$$f_1(x) = 0 \quad \text{for } x > l, \quad (32)$$

$$f_2(x) = \sum_{n=0}^{\infty} b_n P_{2n}^{(1/2, 1/2)}\left(\frac{x}{l}\right) \left(1 - \frac{x^2}{l^2}\right)^{1/2} \quad \text{for } 0 \leq x \leq l, \quad (33)$$

$$f_2(x) = 0 \quad \text{for } x > l, \quad (34)$$

where a_n and b_n are unknown coefficients, $P_n^{(1/2, 1/2)}(x)$ is a Jacobi polynomial [27]. $(1 - (x^2/l^2))^{1/2}$ is the weight function. The weight function can be determined by the singularities of the stress field at the crack tips and relations between the stress fields and the displacement fields.

The Fourier transform of (31)–(34) is [28]

$$\bar{f}_1(s) = \sum_{n=0}^{\infty} a_n G_n^{(1)} \frac{1}{s} J_{2n+2}(sl), \quad G_n^{(1)} = \sqrt{\pi} (-1)^n \frac{\Gamma(2n+2 + (1/2))}{(2n+1)!}, \quad (35)$$

$$\bar{f}_2(s) = \sum_{n=0}^{\infty} b_n G_n^{(2)} \frac{1}{s} J_{2n+1}(sl), \quad G_n^{(2)} = \sqrt{\pi} (-1)^n \frac{\Gamma(2n+1 + (1/2))}{(2n)!}, \quad (36)$$

where $\Gamma(x)$ and $J_n(x)$ are the Gamma and Bessel functions, respectively.

Substituting (35) and (36) into Eqs. (27)–(30), it can be shown that Eqs. (29) and (30) are automatically satisfied. After integration with respect to x in $[0, x]$, Eqs. (27) and (28) reduce to

$$\frac{2}{\pi} \sum_{n=0}^{\infty} \int_0^{\infty} \frac{1}{s^2} [d_1(s) a_n G_n^{(1)} J_{2n+2}(sl) + d_2(s) b_n G_n^{(2)} J_{2n+1}(sl)] \sin(sx) ds = -\sigma_0 x, \quad 0 \leq x \leq l, \quad (37)$$

$$\sum_{n=0}^{\infty} \int_0^{\infty} \frac{1}{s^2} [d_3(s) a_n G_n^{(1)} J_{2n+2}(sl) + d_4(s) b_n G_n^{(2)} J_{2n+1}(sl)] [\cos(sx) - 1] ds = 0, \quad 0 \leq x \leq l. \quad (38)$$

From the relationships [27]

$$\int_0^{\infty} \frac{1}{s} J_n(sa) \sin(bs) ds = \begin{cases} \frac{\sin[n \sin^{-1}(b/a)]}{n} & a > b, \\ \frac{a^n \sin(n\pi/2)}{n[b + \sqrt{b^2 - a^2}]^n} & b > a, \end{cases} \quad (39)$$

$$\int_0^{\infty} \frac{1}{s} J_n(sa) \cos(bs) ds = \begin{cases} \frac{\cos[n \sin^{-1}(b/a)]}{n} & a > b, \\ \frac{a^n \cos(n\pi/2)}{n[b + \sqrt{b^2 - a^2}]^n} & b > a, \end{cases} \quad (40)$$

the semi-infinite integral in Eqs. (37) and (38) can be modified as:

$$\int_0^{\infty} \frac{d_2(s)}{s^2} J_{2n+1}(sl) \sin(sx) ds = \frac{\delta}{2n+1} \sin \left[(2n+1) \sin^{-1} \left(\frac{x}{l} \right) \right] + \int_0^{\infty} \frac{1}{s} \left[\frac{d_2(s)}{s} - \delta \right] J_{2n+1}(sl) \sin(sx) ds, \quad (41)$$

$$\int_0^{\infty} \frac{d_3(s)}{s^2} J_{2n+2}(sl) \cos(sx) ds = \frac{\delta}{2n+2} \cos \left[(2n+2) \sin^{-1} \left(\frac{x}{l} \right) \right] + \int_0^{\infty} \frac{1}{s} \left[\frac{d_3(s)}{s} - \delta \right] J_{2n+2}(sl) \cos(sx) ds, \quad (42)$$

where

$$\lim_{s \rightarrow \infty} \frac{d_1(s)}{s} = \lim_{s \rightarrow \infty} \frac{d_4(s)}{s} = 0, \quad \lim_{s \rightarrow \infty} \frac{d_2(s)}{s} = \lim_{s \rightarrow \infty} \frac{d_3(s)}{s} = \delta = -\frac{2\mu_0}{1+k}. \quad (43)$$

These constants can be obtained by using the Mathematica program and are independent of the gradient parameter β . Also, these constants equal to the ones in the case of the homogeneous material. The multi-valued functions $\lambda_1, \lambda_2, \lambda_3$ and λ_4 , have branch points. We choose the branches such that $\operatorname{Re}(\lambda_1) \geq 0, \operatorname{Re}(\lambda_2) \geq 0, \operatorname{Re}(\lambda_3) \leq 0$ and $\operatorname{Re}(\lambda_4) \leq 0$ on the path of integration. The semi-infinite integral in (37), (38), (41) and (42) can be evaluated directly. Eqs. (37) and (38) can now be solved for the coefficients a_n and b_n by the Schmidt method [24,25]. For briefly, (37) and (38) can be rewritten as

$$\sum_{n=0}^{\infty} a_n E_n^*(x) + \sum_{n=0}^{\infty} b_n F_n^*(x) = U_0(x), \quad 0 \leq x \leq l, \quad (44)$$

$$\sum_{n=0}^{\infty} a_n G_n^*(x) + \sum_{n=0}^{\infty} b_n H_n^*(x) = 0, \quad 0 \leq x \leq l, \quad (45)$$

where $E_n^*(x), F_n^*(x), G_n^*(x)$ and $H_n^*(x)$ and $U_0(x)$ are known functions. a_n and b_n are unknown coefficients.

From Eq. (45), it can be obtained

$$\sum_{n=0}^{\infty} b_n H_n^*(x) = -\sum_{n=0}^{\infty} a_n G_n^*(x). \quad (46)$$

It can now be solved for the coefficients b_n by the Schmidt method. Here the form $-\sum_{n=0}^{\infty} a_n G_n^*(x)$ can be considered as a known function temporarily. A set of functions $P_n(x)$, which satisfy the orthogonality condition

$$\int_0^l P_m(x) P_n(x) dx = N_n \delta_{mn}, \quad N_n = \int_0^l P_n^2(x) dx \quad (47)$$

can be constructed from the function, $H_n^*(x)$, such that

$$P_n(x) = \sum_{i=0}^n \frac{M_{in}}{M_{nn}} H_i^*(x), \quad (48)$$

where M_{ij} is the cofactor of the element d_{ij} of D_n , which is defined as

$$D_n = \begin{bmatrix} d_{00}, d_{01}, d_{02}, \dots, d_{0n} \\ d_{10}, d_{11}, d_{12}, \dots, d_{1n} \\ d_{20}, d_{21}, d_{22}, \dots, d_{2n} \\ \dots \\ \dots \\ \dots \\ d_{n0}, d_{n1}, d_{n2}, \dots, d_{nn} \end{bmatrix}, \quad d_{ij} = \int_0^l H_i^*(x) H_j^*(x) dx. \quad (49)$$

Using (46)–(49), we obtain

$$b_n = \sum_{j=n}^{\infty} q_j \frac{M_{nj}}{M_{jj}} \quad \text{with} \quad q_j = -\sum_{i=0}^{\infty} a_i \frac{1}{N_j} \int_0^l G_i^*(x) P_j(x) dx. \quad (50)$$

Hence, it can be rewritten as

$$b_n = \sum_{i=0}^{\infty} a_i K_{in}^*, \quad K_{in}^* = - \sum_{j=n}^{\infty} \frac{M_{nj}}{N_j M_{jj}} \int_0^l G_i^*(x) P_j(x) dx. \quad (51)$$

Substituting (51) in Eq. (44), it can be obtained

$$\sum_{n=0}^{\infty} a_n Y_n^*(x) = U_0(x), \quad Y_n^*(x) = E_n^*(x) + \sum_{i=0}^{\infty} K_{ni}^* F_i^*(x). \quad (52)$$

Hence, it can now be solved for the coefficients a_n by the Schmidt method again as mentioned above. With the aid of (51), the coefficients b_n can be obtained.

5. Intensity factors

The coefficients a_n and b_n are known, so that the entire stress field can be obtained. However, in fracture mechanics, it is important to determine stresses $\sigma_y^{(2)}$ and $\tau_{xy}^{(2)}$ in the vicinity of the crack tips. $\sigma_y^{(2)}$ and $\tau_{xy}^{(2)}$ along the crack line can be expressed as:

$$\begin{aligned} \sigma_y^{(2)}(x, 0, t) &= \frac{2}{\pi} \sum_{n=0}^{\infty} \int_0^{\infty} \frac{1}{s} [d_1(s) a_n G_n^{(1)} J_{2n+2}(sl) + d_2(s) b_n G_n^{(2)} J_{2n+1}(sl)] \cos(sx) ds \\ &= \frac{2}{\pi} \sum_{n=0}^{\infty} \int_0^{\infty} \left\{ \frac{d_1(s)}{s} a_n G_n^{(1)} J_{2n+2}(sl) + \left[\left(\frac{d_2(s)}{s} - \delta \right) + \delta \right] b_n G_n^{(2)} J_{2n+1}(sl) \right\} \cos(sx) ds, \end{aligned} \quad (53)$$

$$\begin{aligned} \tau_{xy}^{(2)}(x, 0, t) &= \frac{2}{\pi} \sum_{n=0}^{\infty} \int_0^{\infty} \frac{1}{s} [d_3(s) a_n G_n^{(1)} J_{2n+2}(sl) + d_4(s) b_n G_n^{(2)} J_{2n+1}(sl)] \sin(sx) ds \\ &= \frac{2}{\pi} \sum_{n=0}^{\infty} \int_0^{\infty} \left\{ \left[\left(\frac{d_3(s)}{s} - \delta \right) + \delta \right] a_n G_n^{(1)} J_{2n+2}(sl) + \frac{d_4(s)}{s} b_n G_n^{(2)} J_{2n+1}(sl) \right\} \sin(sx) ds. \end{aligned} \quad (54)$$

An examination of (53) and (54) shows that, the singular part of the stress field can be obtained from the relationships as follows [27]:

$$\begin{aligned} \int_0^{\infty} J_n(sa) \cos(bs) ds &= \begin{cases} \frac{\cos[n \sin^{-1}(b/a)]}{\sqrt{a^2 - b^2}} & a > b, \\ -\frac{a^n \sin(n\pi/2)}{\sqrt{b^2 - a^2} [b + \sqrt{b^2 - a^2}]^n} & b > a, \end{cases} \\ \int_0^{\infty} J_n(sa) \sin(bs) ds &= \begin{cases} \frac{\sin[n \sin^{-1}(b/a)]}{\sqrt{a^2 - b^2}} & a > b, \\ \frac{a^n \cos(n\pi/2)}{\sqrt{b^2 - a^2} [b + \sqrt{b^2 - a^2}]^n} & b > a. \end{cases} \end{aligned}$$

The singular part of the stress field can be expressed, respectively, as follows ($l < x$):

$$\sigma = -\frac{2\delta}{\pi} \sum_{n=0}^{\infty} b_n G_n^{(2)} H_n^{(1)}(x), \quad (55)$$

$$\tau = \frac{2\delta}{\pi} \sum_{n=0}^{\infty} a_n G_n^{(1)} H_n^{(2)}(x), \quad (56)$$

where

$$H_n^{(1)}(x) = \frac{(-1)^n l^{2n+1}}{\sqrt{x^2 - l^2} \left[x + \sqrt{x^2 - l^2} \right]^{2n+1}}, \quad H_n^{(2)}(x) = \frac{(-1)^{n+1} l^{2n+2}}{\sqrt{x^2 - l^2} \left[x + \sqrt{x^2 - l^2} \right]^{2n+2}}.$$

We obtain the stress intensity factors K_I and K_{II} as follows:

$$K_I = \lim_{x \rightarrow l^+} \sqrt{2(x-l)}\sigma = -\frac{2\delta}{\sqrt{\pi l}} \sum_{n=0}^{\infty} b_n \frac{\Gamma(2n+1+(1/2))}{(2n)!}, \quad (57)$$

$$K_{II} = \lim_{x \rightarrow l^+} \sqrt{2(x-l)}\tau = -\frac{2\delta}{\sqrt{\pi l}} \sum_{n=0}^{\infty} a_n \frac{\Gamma(2n+2+(1/2))}{(2n+1)!}. \quad (58)$$

6. Numerical calculations and discussion

To check the numerical accuracy of the Schmidt method, the results of this paper for $cl = 0$ with the corresponding static results are given in Table 1. It can be seen that the results of this paper for $cl = 0$ are very close to Konda's results [31]. The stress intensity factors of this paper for $\beta l = 0.01$ ($\mu_0 = 77 \times 10^9 \text{ N/m}^2$, $\rho_0 = 7800 \text{ kg/m}^3$ and $\nu = 0.28$) are $K_I(l)/(\sigma_0\sqrt{l}) = 1.00035$ ($cl = 0.005$), $K_{II}(l)/(\sigma_0\sqrt{l}) = 0.00250101$ ($cl = 0.005$), $K_I(l)/(\sigma_0\sqrt{l}) = 1.00075$ ($cl = 0.01$), $K_{II}(l)/(\sigma_0\sqrt{l}) = 0.00250282$ ($cl = 0.01$), $K_I(l)/(\sigma_0\sqrt{l}) = 1.0233$ ($cl = 0.105$), $K_{II}(l)/(\sigma_0\sqrt{l}) = 0.00260795$ ($cl = 0.105$), $K_I(l)/(\sigma_0\sqrt{l}) = 1.0461$ ($cl = 0.205$), $K_{II}(l)/(\sigma_0\sqrt{l}) = 0.00257766$ ($cl = 0.205$). It can be obtained that the results of this paper for 'small' βl approximate to van der Hijden and Neerhoff's results [32].

Where $K_I^*/(\sigma_0\sqrt{l})$ and $K_{II}^*/(\sigma_0\sqrt{l})$ are Konda's results [31]. $K_I/(\sigma_0\sqrt{l})$ and $K_{II}/(\sigma_0\sqrt{l})$ are this paper's results. Finally, the values of

$$\frac{2 \left[\sum_{n=0}^9 a_n E_n^*(x) + \sum_{n=0}^9 b_n F_n^*(x) \right]}{\pi \sigma_0}$$

and $U_0(x)/\sigma_0$ are given in Table 2 for $\beta l = 0.3$, $cl = 0.3$. In Table 3, the values of the coefficients a_n and b_n are given for $\beta l = 0.3$ and $cl = 0.3$.

From [24,25,29,30] and the above discussion, it can be seen that the Schmidt method is performed satisfactorily if the first 10 terms of infinite series to Eqs. (44) and (45) are retained. The behavior of the sum of the series keeps steady with the increasing number of terms in (44) and (45). At $-l \leq x \leq l$, $y = 0$, it can be obtained that $\sigma_y^{(2)}/\sigma_0$ is very close to negative unity. Hence, the solution of this paper can also be proved to satisfy the boundary conditions (2). The material constants of the functionally graded materials are assumed as $\mu_0 = 77.0 \text{ e}^{\beta y} (\times 10^9 \text{ N/m}^2)$,

Table 1

Verification of the model ($cl = 0$, $\mu_0 = 77 \times 10^9 \text{ N/m}^2$, $\rho_0 = 7800 \text{ kg/m}^3$ and $\nu = 0.28$)

βl	$K_I^*(l)/(\sigma_0\sqrt{l})$	$K_I(l)/(\sigma_0\sqrt{l})$	$K_{II}^*(l)/(\sigma_0\sqrt{l})$	$K_{II}(l)/(\sigma_0\sqrt{l})$
0.1	1.0085	1.00855	0.0262	0.0250085
0.25	1.036	1.03657	0.065	0.0625919
0.5	1.101	1.10147	0.129	0.125660
1.0	1.258	1.26049	0.263	0.254639

Table 2

Values of $2 \left[\sum_{n=0}^9 a_n E_n^*(x) + \sum_{n=0}^9 b_n F_n^*(x) \right] / \pi \sigma_0$ and $U_0(x)/\sigma_0$ for $\beta l = 0.3$, $cl = 0.3$ ($\mu_0 = 77 \times 10^9 \text{ N/m}^2$, $\rho_0 = 7800 \text{ kg/m}^3$ and $\nu = 0.28$)

x	$2 \left[\sum_{n=0}^9 a_n E_n^*(x) + \sum_{n=0}^9 b_n F_n^*(x) \right] / \pi \sigma_0$		$U_0(x)/\sigma_0$
	Real part	Imaginary part	
0.1	−0.10003	0.132925D−07	−0.1
0.2	−0.20017	0.130927D−07	−0.2
0.3	−0.30010	0.122412D−07	−0.3
0.4	−0.40005	0.963570D−08	−0.4
0.5	−0.50027	0.691544D−08	−0.5
0.6	−0.60025	0.416302D−08	−0.6
0.7	−0.70034	0.538018D−09	−0.7
0.8	−0.80056	0.384785D−08	−0.8
0.9	−0.90078	0.699468D−08	−0.9

Table 3

Values of a_n and b_n for $\beta l = 0.3$, $cl = 0.3$ ($\mu_0 = 77 \times 10^9 \text{ N/m}^2$, $\rho_0 = 7800 \text{ kg/m}^3$ and $\nu = 0.28$)

n	$2a_n/\pi\sigma_0$		$2b_n/\pi\sigma_0$	
	Real part	Imaginary part	Real part	Imaginary part
0	−0.780367D−03	−0.788443D−04	−0.146674D−01	−0.105707D−02
1	0.632474D−04	0.547235D−05	0.196768D−03	0.163178D−04
2	−0.434266D−05	−0.321604D−06	−0.770882D−04	−0.652842D−05
3	−0.147576D−05	−0.968702D−07	0.220698D−04	0.216505D−06
4	0.107723D−05	0.770777D−08	0.506925D−05	0.101778D−06
5	0.253876D−06	0.173066D−08	−0.481774D−06	−0.538656D−07
6	0.537253D−07	0.423652D−09	−0.740552D−07	−0.556831D−08
7	−0.245545D−07	−0.400122D−09	0.420578D−08	0.206401D−08
8	0.303421D−08	0.320341D−09	0.102324D−08	0.1213728D−08
9	0.454273D−09	0.103023D−09	−0.461321D−09	−0.433652D−09

$\rho_0 = 7800 \text{ kg/m}^3$ and $\nu = 0.28$, respectively. The dimensionless stress intensity factors $K/\sigma_0\sqrt{l}$ are calculated numerically. The results of this paper are shown in Figs. 2–4. From the results, the following observations are very significant:

- (i) The aim of this paper is just to give an approach to solve the dynamic fracture problem in the functionally graded materials subjected to the harmonic stress waves. In this paper, the unknown variables of dual integral equations are the displacement jumps across the crack surfaces. From the results, it can be obtained the stress

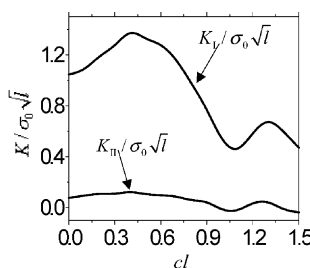


Fig. 2. The stress intensity factor versus cl ($c = \omega\sqrt{\rho_0/\mu_0}$) for $\beta l = 0.3$, $\mu_0 = 77 \times 10^9 \text{ N/m}^2$, $\rho_0 = 7800 \text{ kg/m}^3$ and $\nu = 0.28$.

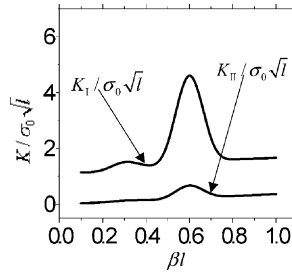


Fig. 3. The stress intensity factor versus βl for $cl = 0.3$ ($c = \omega\sqrt{\rho_0/\mu_0}$), $\mu_0 = 77 \times 10^9 \text{ N/m}^2$, $\rho_0 = 7800 \text{ kg/m}^3$ and $\nu = 0.28$.

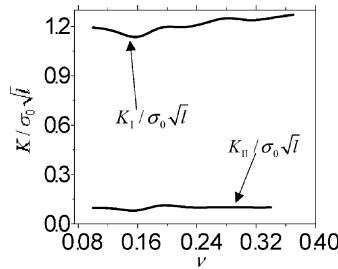


Fig. 4. The stress intensity factor versus ν for $cl = 0.3$ ($c = \omega\sqrt{\rho_0/\mu_0}$), $\beta l = 0.3$, $\mu_0 = 77 \times 10^9 \text{ N/m}^2$ and $\rho_0 = 7800 \text{ kg/m}^3$.

intensity factors are dependent on the crack length, the material parameters and the circular frequency of the incident wave.

- (ii) The normalization stress intensity factors $K_I/(\sigma_0\sqrt{l})$ and $K_{II}/(\sigma_0\sqrt{l})$ increase with the increase in the circular frequency of the incident wave until reaching a peak at $cl \approx 0.4$. Then they come to decrease until reaching a minimum at $cl \approx 1.05$ as shown in Fig. 2.
- (iii) As shown in Fig. 3, the normalization stress intensity factor $K_I/(\sigma_0\sqrt{l})$ changes slowly with increasing of βl for $\beta l < 0.48$. However, the normalization stress intensity factor $K_I/(\sigma_0\sqrt{l})$ increases quickly with increasing of βl until reaching a peak at $\beta l \approx 0.6$, then it comes to decrease for $\beta l > 0.48$. The normalization stress intensity factor $K_{II}/(\sigma_0\sqrt{l})$ has the same behavior as the normalization stress intensity factor $K_I/(\sigma_0\sqrt{l})$ versus βl .
- (iv) As shown in Fig. 4, it can be obtained that the variation of the Poisson's ratio ν within a practical range has the rather insignificant influence on the normalization stress intensity factors $K_I/(\sigma_0\sqrt{l})$ and $K_{II}/(\sigma_0\sqrt{l})$. A similar behavior has been obtained in [5–7].
- (v) The normalization shear stress intensity factor $K_{II}/(\sigma_0\sqrt{l})$ is smaller than the normalization stress intensity factor $K_I/(\sigma_0\sqrt{l})$.

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Appendix A

$$\begin{aligned}
 [X_1] &= \mu_0 \begin{bmatrix} \frac{-(k+1)m_3(s)\lambda_3 + s(3-k)}{k-1} & \frac{-(k+1)m_4(s)\lambda_4 + s(3-k)}{k-1} \\ -\lambda_3 - sm_3(s) & -\lambda_4 - sm_4(s) \end{bmatrix}, \\
 [X_2] &= \mu_0 \begin{bmatrix} \frac{-(k+1)m_1(s)\lambda_1 + s(3-k)}{k-1} & \frac{-(k+1)m_2(s)\lambda_2 + s(3-k)}{k-1} \\ -\lambda_1 - sm_1(s) & -\lambda_2 - sm_2(s) \end{bmatrix}, \\
 [X_3] &= \begin{bmatrix} 1 & 1 \\ m_1(s) & m_2(s) \end{bmatrix}, \quad [X_4] = \begin{bmatrix} 1 & 1 \\ m_3(s) & m_4(s) \end{bmatrix}, \\
 [Y_1] &= [X_3] - [X_4][X_1]^{-1}[X_2], \quad [Y_2] = [X_2][Y_1]^{-1} = \begin{bmatrix} d_1(s) & d_2(s) \\ d_3(s) & d_4(s) \end{bmatrix}.
 \end{aligned}$$

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